THE DEGREE OF THE THIRD SECANT VARIETY OF A SMOOTH CURVE OF GENUS 2

ANDREA HOFMANN

Abstract. We give a new method of computation of the degree of the third secant variety $\mathrm{Sec}_3(C)$ of a smooth curve $C \subseteq \mathbf{P}^{d-2}$ of genus 2 and degree $d \geq 8$, using the presentation of $\mathrm{Sec}_3(C)$ as the union of all scrolls that are defined via a g^1_3 on C.

1. Introduction

Berzolari's formula from 1895 (cf. [2], Section 4) computes the number of trisecant lines to a smooth curve of genus g and degree d in $\mathbf{P}^4$, in the case this number is finite. This number is equal to $\binom{d-2}{3} - g(d-4)$.

In this paper C denotes a smooth and irreducible curve of genus 2 and degree $d \geq 8$ embedded in $\mathbf{P}^{d-2}$.

The third secant variety of C, $\mathrm{Sec}_3(C)$, is defined as the closure of the union of all trisecant planes to C:

$$\mathrm{Sec}_3(C) = \overline{\bigcup_{D \in C_3} \mathrm{span}(D)},$$

where $C_3 := (C \times C \times C)/S_3$ parametrizes effective divisors of degree 3 on C, and $\mathrm{span}(D)$ denotes the plane spanned by the three points in D.

The dimension of $\mathrm{Sec}_3(C)$ is equal to $\dim(C_3) + \dim(\mathrm{span}(D)) = 5$, i.e. in order to find the degree of $\mathrm{Sec}_3(C)$ we have to intersect with five general hyperplanes.

Let now V denote the intersection of five general hyperplanes in $\mathbf{P}^{d-2}$, i.e. V is a general space of codimension 5 in $\mathbf{P}^{d-2}$. Since $\dim(\mathrm{Sec}_2(C)) = 3$ and $\mathrm{codim}(V) = 5$, V and $\mathrm{Sec}_2(C)$ do not intersect. This implies that V cannot intersect any trisecant plane to C in a line, since every trisecant plane to C contains three lines in $\mathrm{Sec}_2(C)$, and so if V intersects a trisecant plane to C in a line L, then L intersects at least one of those lines in $\mathrm{Sec}_2(C)$ in a point which obviously lies in $\mathrm{Sec}_2(C)$.

Projecting from V to $\mathbf{P}^4$ gives us the equality of the degree of $\mathrm{Sec}_3(C)$ and the number of trisecant lines to a curve $C \subseteq \mathbf{P}^4$ of genus 2 and the same degree d in the following way: Since V was chosen to be a general space of codimension 5, V does not intersect the curve C, and thus the image of C under the projection from V is a curve of degree d and genus 2 in $\mathbf{P}^4$. Moreover, the fact that V does not intersect $\mathrm{Sec}_2(C)$ implies that the image curve is smooth as well.

A trisecant plane to $C \subseteq \mathbf{P}^{d-2}$ which intersects V in one point projects down to a trisecant line to the image curve in $\mathbf{P}^4$.
Summarizing, the number of trisecant planes to $C \subseteq \mathbf{P}^{d-2}$ that intersect V in one point is equal to the number of trisecant lines to the image curve in $\mathbf{P}^4$, and thus it follows that the degree of $\mathrm{Sec}_3(C)$ is equal to the number of trisecant lines to the image curve in $\mathbf{P}^4$.
Consequently, the degree of $\mathrm{Sec}_3(C)$ is equal to $\binom{d-2}{3} - 2(d-4)$, and our motivation is now to compute the degree of $\mathrm{Sec}_3(C)$ in a different way, identifying $\mathrm{Sec}_3(C)$ as the union of all scrolls defined via a g^1_3 on C.
Any abstract curve C of genus 2 can be embedded as a smooth curve of degree $d \geq 5$ into $\mathbf{P}^{d-2}$.
In this paper we restrict ourselves to the case $d \geq 8$, since for a curve C of genus 2 and degree $d = 6$ or $d = 7$, although Berzolari's formula of course being valid, taking Berzolari's formula to compute the degree of $\mathrm{Sec}_3(C)$ does not make sense, since for these values of d the third secant variety $\mathrm{Sec}_3(C)$ is equal to the ambient space $\mathbf{P}^{d-2}$. For $d = 5$ the following holds: There are infinitely many trisecant lines to a curve $C \subseteq \mathbf{P}^3$ of genus 2 and degree 5, since C lies on a quadric on which there exists a one-dimensional family of lines that each intersects C in three points.

2. Preliminaries

Let C be a smooth curve of genus 2 and degree $d \geq 8$ embedded in projective space $\mathbf{P}^{d-2}$. For each g^1_3 on C, which we denote by $|D|$, we set

$$V_{|D|} := \overline{\bigcup_{D' \in |D|} \mathrm{span}(D')},$$

where $\mathrm{span}\, D'$ denotes the plane spanned by the three points in $|D|$.
Each $V_{|D|}$ is a threedimensional rational normal scroll. (For general theory about rational normal scrolls we refer to [3].)
We set $G^1_3(C) := \{g^1_3\text{'s on } C\}$. Since our aim is to identify $\mathrm{Sec}_3(C) = \bigcup_{|D| \in G^1_3(C)} V_{|D|}$, we want to find the dimension of $\bigcup_{|D| \in G^1_3(C)} V_{|D|}$, and for this purpose we need the dimension of the family $G^1_3(C)$, which we will now compute:

Proposition 2.1. *Let C be a curve of genus 2. The family $G^1_3(C) = \{g^1_3$'s on $C\}$ is two-dimensional.*

Proof. If D is a divisor of degree 3 on C, then $h^0(\mathcal{O}_C(D)) = 2$ by the Riemann-Roch theorem for curves (see e.g. [5], Thm. 1.3 in Chapter IV.1), i.e. each linear system $|D|$ of degree 3 is a g^1_3 on C. The set of all effective divisors of degree 3 on C is given by $C_3 := (C \times C \times C)/S_3$, where S_3 denotes the symmetric group on 3 letters. The dimension of this family is equal to 3, and since each linear system $|D|$ of degree 3 on C has dimension 1, as shown above, the family of g^1_3's on C has to be two-dimensional. $\square$

We obtain that the dimension of $\bigcup_{|D|\in G^1_3(C)} V_{|D|}$ is equal to 5, which is also the dimension of $\mathrm{Sec}_3(C)$, as we have seen in the introduction, and since each scroll $V_{|D|}$ obviously is contained in $\mathrm{Sec}_3(C)$, we obtain equality:

$$\mathrm{Sec}_3(C) = \bigcup_{|D|\in G^1_3(C)} V_{|D|}.$$

For an integer $k \geq 0$ we denote by $\mathrm{Pic}^k(C)$ the set of all line bundles of degree k on C modulo isomorphism. In this paper we will consider $k = 0$ and $k = 3$.
We use the definition of the Jacobian variety of C, $\mathrm{Jac}(C)$, as in [6], namely that $\mathrm{Jac}(C)$ is defined as the abelian variety that represents the functor $T \to Pic^0(C/T)$ from schemes over the base field k to abelian groups (cf. [6], Theorem 1.1).
By fixing a divisor D_0 of degree 3 we obtain an isomorphism

$$\mu : \mathrm{Pic}^0(C) \to \mathrm{Pic}^3(C),$$
$$[\mathcal{O}_C(D)] \mapsto [\mathcal{O}_C(D + D_0)].$$

Hence $\mathrm{Pic}^3(C)$ is isomorphic to the Jacobian variety $\mathrm{Jac}(C)$.
Fixing a point P_0 on C gives an embedding

$$\nu : C \to \mathrm{Jac}(C),$$
$$R \mapsto [\mathcal{O}_C(R - P_0)].$$

The dimension of $\mathrm{Jac}(C)$ is equal to the genus of C, which is equal to 2. Hence $\mathrm{Jac}(C)$ is an abelian surface. The theta divisor Θ on $\mathrm{Jac}(C)$ is the image of C under the above map ν.
For fixed points P and Q on C we define

$$\Theta_{P,Q} := \{[\mathcal{O}_C(P + Q + R)] | R \in C\}.$$

$\Theta_{P,Q}$ is a divisor on $\mathrm{Pic}^3(C)$, and using the above isomorphism μ with $D_0 = P + Q + P_0$ we see that the divisor $\Theta_{P,Q}$ is isomorphic to Θ. It is this $\Theta_{P,Q}$ we will use in Sections 4 and 5 when we consider Θ on $\mathrm{Pic}^3(C)$.

Proposition 2.2. *The divisor Θ has self-intersection $\Theta^2 = 2$.*

Proof. Choose points P, P', Q_1 and Q_2, $Q_1 \neq Q_2$, on C such that $P + P'$ is a divisor in the canonical system on C, $|K_C|$, and such that $Q_1 + Q_2$ is not a divisor in $|K_C|$. There exist points Q'_1 and Q'_2 on C such that $Q_1 + Q'_1 \in |K_C|$ and $Q_2 + Q'_2 \in |K_C|$. We obtain the following:

$$\begin{aligned} \Theta^2 &= \Theta_{P,Q_1}.\Theta_{P',Q_2} \\ &= \#\{[\mathcal{O}_C(Q_1 + Q_2 + R)] | R \in \{Q'_1, Q'_2\}\} \\ &= 2. \end{aligned}$$

□

Consider now the following projections:

$$\begin{tikzcd} & C \times \mathrm{Pic}^3(C) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C & & \mathrm{Pic}^3(C) \end{tikzcd} \qquad \begin{tikzcd} & C \times C_3 \arrow[dl] \arrow[dr, "\pi"] & \\ C & & C_3 \end{tikzcd}$$

Let P be a point on C such that $2P$ is a divisor in the canonical system $|K_C|$, and set $f := p^*(P)$.
In the rest of this paper we will use the notation P and f both as varieties and as classes.
As before, we define C_3 to be the threedimensional family of all effective divisors of degree 3 on C.
Let Δ be the universal divisor on $C \times C_3$, i.e. $\Delta|_{C\times\{D\}} \cong D$ for all $D \in C_3$.
For any point Q on C set $X_Q := \{D \in C_3 | Q \in D\}$, which is a divisor on C_3.
Finally, let $u : C_3 \to \mathrm{Pic}^3(C)$ be the map given by $u(D) := [\mathcal{O}_C(D)]$.
Now we are able to define a line bundle $\mathcal{L}$ on $C \times \mathrm{Pic}^3(C)$ which turns out to be a Poincaré line bundle. In Section 5 we will compute the degree of $\mathrm{Sec}_3(C)$ by identifying $\mathrm{Sec}_3(C)$ as a degeneracy locus of a map of vector bundles involving this Poincaré line bundle $\mathcal{L}$.

3. The Poincaré line bundle $\mathcal{L}$

We will first give the definition of a Poincaré line bundle:

Definition 3.1. *A* Poincaré line bundle of degree k *is a line bundle* $\mathcal{L}$ *on* $C \times \mathrm{Pic}^k(C)$ *such that* $\mathcal{L}|_{C\times[\mathcal{O}_C(D)]} \cong \mathcal{O}_C(D)$ *for all points* $[\mathcal{O}_C(D)]$ *in* $\mathrm{Pic}^k(C)$.

Set $\mathcal{L} := (1 \times u)_*(\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q)))$. $\mathcal{L}$ is a Poincaré line bundle of degree 3 (cf. [1], Chapter IV, §2, p. 167).
Let $|H|$ be the linear system of degree d that embeds C into projective space $\mathbf{P}^{d-2}$. Set $\mathcal{H} := q_*(\mathcal{L})$ and $\mathcal{G} := q_*(p^*\mathcal{O}_C(H) \otimes \mathcal{L}^{-1})$.
Since the fiber of $\mathcal{H}$ over a point $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ is equal to $H^0(\mathcal{O}_C(D))$, the rank of $\mathcal{H}$ is equal to $h^0(\mathcal{O}_C(D)) = 2$, and since the fiber of $\mathcal{G}$ over a point $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ is equal to $H^0(\mathcal{O}_C(H-D))$, the rank of $\mathcal{G}$ is equal to $h^0(\mathcal{O}_C(H-D)) = d-4$.
We will use these vector bundles $\mathcal{H}$ and $\mathcal{G}$ in Section 5 to define a map of vector bundles which degeneracy locus is equal to the third secant variety of C, $\mathrm{Sec}_3(C)$. We will need the Chern classes of $\mathcal{H}$ and $\mathcal{G}$, and for this purpose we need the Chern classes of $\mathcal{L}$. We will find all of these Chern classes in the next section.

4. The Chern classes of $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{G}$

In this section we will find the Chern classes of $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{G}$ as defined in Section 3.

4.1. **The Chern class of $\mathcal{L}$.** By [1], Chapter VIII, §2 (pp. 333-336) we obtain that the first Chern class of $\mathcal{L}$ is equal to $c_1(\mathcal{L}) = 3f + \gamma$, where γ is the diagonal component of $c_1(\mathcal{L})$ in the term $H^1(C) \otimes H^1(\mathrm{Pic}^3(C))$ of the Künneth decomposition

$$\begin{aligned} H^2(C \times \mathrm{Pic}^3(C)) &= (H^2(C) \otimes H^0(\mathrm{Pic}^3(C))) \\ &\oplus (H^1(C) \otimes H^1(\mathrm{Pic}^3(C))) \\ &\oplus (H^0(C) \otimes H^2(\mathrm{Pic}^3(C))). \end{aligned}$$

The following is satisfied: $\gamma^2 = -2f.q^*(\Theta)$, $\gamma^3 = f.\gamma = 0$, where now Θ on $\mathrm{Pic}^3(C)$ is equal to $\Theta_{P,Q}$ as defined in Section 2.
Thus for the Chern character of $\mathcal{L}$ we obtain:

$$\mathrm{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + 3f + \gamma - f.q^*(\Theta).$$

4.2. **The Chern classes of $\mathcal{H}$.** Recall that we had defined $\mathcal{H} := q_*\mathcal{L}$. The Chern character of $\mathcal{H}$ we obtain by the Grothendieck-Riemann-Roch Theorem (cf. [4], Thm. 15.2):

$$\mathrm{ch}(q_*(\mathcal{L})).\,\mathrm{td}(\mathrm{Pic}^3(C)) = q_*(\mathrm{ch}(\mathcal{L}).\,\mathrm{td}(C \times \mathrm{Pic}^3(C))).$$

Before we can continue our computation of $\mathrm{ch}(\mathcal{H})$ we need some Todd classes and pushforwards.

Definition 4.1. *(cf. [4], Example 3.2.4) The Todd class of a vector bundle E of rank r on a variety X is defined as*

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{\alpha_i}},$$

where $\alpha_1, \ldots, \alpha_r$ are the Chern roots of E.
If Y is a variety, then by $\mathrm{td}(Y)$ we denote $\mathrm{td}(T_Y)$, the Todd class of the tangent bundle of Y.

We will need Todd classes only in the cases when the dimension of X is equal to 1 or 2. In these cases $c_i(E) = 0$ for $i \geq 3$, and expanding the above product yields:

$$\mathrm{td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)).$$

Lemma 4.2. *We have the following Todd classes:*
(1) $\mathrm{td}(\mathrm{Pic}^3(C)) = 1$.
(2) $\mathrm{td}(C) = 1 - P$.
(3) $\mathrm{td}(C \times \mathrm{Pic}^3(C)) = 1 - f$.

Proof.
(1) Since $\mathrm{Pic}^3(C) \cong \mathrm{Jac}(C)$ is an abelian variety, we have $K_{\mathrm{Pic}^3(C)} = 0$ and thus also $c_1(T_{\mathrm{Pic}^3(C)}) = 0$.
(2) $\mathrm{td}(C) = 1 + \frac{1}{2}c_1(T_C) = 1 - \frac{1}{2}[K_C] = 1 - P$.

(3) $\operatorname{td}(C \times \operatorname{Pic}^3(C)) = \operatorname{td}(p^*(C)).\operatorname{td}(q^* \operatorname{Pic}^3(C)) = 1 - f.$ □

Lemma 4.3. *We have the following pushforwards:*

(1) $q_*(1) = 0.$
(2) $q_*(f) = 1.$
(3) $q_*(\gamma) = 0.$

Proof.

(1) $q_*(1) = q_*([C \times \operatorname{Pic}^3(C)]) = 0$, since $\dim(q(C \times \operatorname{Pic}^3(C))) = \dim(\operatorname{Pic}^3(C)) = 2 < 3 = \dim(C \times \operatorname{Pic}^3(C))$.

(2) Since $q(f) = \operatorname{Pic}^3(C)$ has the same dimension as f, we have $q_*(f) = a[\operatorname{Pic}^3(C)]$ for a positive integer a. By the projection formula (cf. [4], Prop. 2.5(c)) we obtain for every point $[\mathcal{O}_C(D_0)] \in \operatorname{Pic}^3(C)$:

$$\begin{aligned} a &= q_*(f).[\mathcal{O}_C(D_0)] = q_*(f.q^*[\mathcal{O}_C(D_0)]) = f.q^*[\mathcal{O}_C(D_0)] \\ &= [P \times \operatorname{Pic}^3(C)].[C \times \mathcal{O}_C(D_0)] = 1, \end{aligned}$$

where we could use the equality $q_*(f.q^*[\mathcal{O}_C(D_0)]) = f.q^*[\mathcal{O}_C(D_0)]$, since $f.q^*[\mathcal{O}_C(D_0)]$ is 0-dimensional.

(3) Since γ is of codimension 1 on $C \times \operatorname{Pic}^3(C)$, $q_*(\gamma) = a[\operatorname{Pic}^3(C)]$ for some non-negative integer a. By the projection formula we have for every point $[\mathcal{O}_C(D_0)] \in \operatorname{Pic}^3(C)$:

$$\begin{aligned} a &= q_*(\gamma).[\mathcal{O}_C(D_0)] = q_*(\gamma.q^*[\mathcal{O}_C(D_0)]) = \gamma.q^*[\mathcal{O}_C(D_0)] \\ &= c_1(\mathcal{L}).q^*[\mathcal{O}_C(D_0)] - q_*(3f) = 3 - 3 = 0, \end{aligned}$$

where, analogously to (2), we could use the equality $q_*(\gamma.q^*[\mathcal{O}_C(D_0)]) = \gamma.q^*[\mathcal{O}_C(D_0)]$ since $\gamma.q^*[\mathcal{O}_C(D_0)]$ is 0-dimensional.

□

Now, by Lemma 4.2 we obtain:

$$\begin{aligned} \operatorname{ch}(\mathcal{H}) &= \operatorname{ch}(q_*(\mathcal{L})) \\ &= q_*(\operatorname{ch}(\mathcal{L}).(1-f)) = q_*((1 + 3f + \gamma - f.q^*(\Theta)).(1-f)) \\ &= q_*(1 + 2f + \gamma - f.q^*(\Theta)). \end{aligned}$$

By Lemma 4.3 and the projection formula we can conclude:

$$\operatorname{ch}(\mathcal{H}) = 2 - q_*(f).\Theta = 2 - \Theta.$$

Consequently we obtain for the Chern polynomial of $\mathcal{H}$:

$$c_t(\mathcal{H}) = e^{-\Theta t}. \tag{1}$$

4.3. **The Chern classes of $\mathcal{G}$.** Now we want to find the Chern classes of the vector bundle $\mathcal{G} := q_*(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1}))$, where $|H|$ denotes the linear system of degree d that embeds C into projective space. In order to do so we use again the Grothendieck-Riemann-Roch formula:

$$\mathrm{ch}(q_*(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1}))).\,\mathrm{td}(\mathrm{Pic}^3(C)) = q_*(\mathrm{ch}(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1})).\,\mathrm{td}(C \times \mathrm{Pic}^3(C))).$$

By Lemma 4.2, Lemma 4.3 and the projection formula we obtain

$$\begin{aligned} \mathrm{ch}(\mathcal{G}) &= q_*(\mathrm{ch}(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1}).(1-f)) \\ &= q_*(p^*(\mathrm{ch}(\mathcal{O}_C(H))).\,\mathrm{ch}(\mathcal{L}^{-1}).(1-f)) \\ &= q_*(1+p^*(H)).(1-3f-\gamma-f.q^*(\Theta)).(1-f)) \\ &= q_*((1+df).(1-4f-\gamma-f.q^*(\Theta))) \\ &= q_*(1+(d-4)f-\gamma-f.q^*(\Theta)) \\ &= d-4-\Theta. \end{aligned}$$

This yields for the Chern polynomial of $\mathcal{G}$:

$$c_t(\mathcal{G}) = e^{-\Theta t}. \tag{2}$$

5. The degree of $\mathrm{Sec}_3(C)$

Set $E := \mathcal{G} \boxtimes \mathcal{O}_{\mathbf{P}^{d-2}}(-1)$ and $F := \mathcal{H}^* \boxtimes \mathcal{O}_{\mathbf{P}^{d-2}}$. These are two vector bundles on $\mathrm{Pic}^3(C) \times \mathbf{P}^{d-2}$. The rank of E is equal to $d-4$, and the rank of F is equal to 2.
The multiplication of fibers

$$H^0(\mathcal{O}_C(H-D)) \otimes H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_C(H))$$

induces a map of vector bundles $\Phi : E \to F$. Set

$$X_1 := X_1(\Phi) := \{x \in \mathrm{Pic}^3(C) \times \mathbf{P}^{d-2} |\, \mathrm{rk}(\Phi_x) \leq 1\}$$

Consider the two projections

$$\begin{tikzcd} & \mathrm{Pic}^3(C) \times \mathbf{P}^{d-2} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Pic}^3(C) & & \mathbf{P}^{d-2}. \end{tikzcd}$$

We have the following:

(i) Over every point $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ the fiber of $p_1|X_1$ is a 3-dimensional rational normal scroll $V_{|D|} \subseteq [\mathcal{O}_C(D)] \times \mathbf{P}^{d-2} \cong \mathbf{P}^{d-2}$.

(ii) The image of such a fiber under the projection p_2 is thus the rational normal scroll $V_{|D|}$ in $\mathbf{P}^{d-2}$.

(iii) Consequently, $p_2(X_1)$ is the union of all $g^1_3(C)$-scrolls $V_{|D|}$ in $\mathbf{P}^{d-2}$ which again is equal to $\mathrm{Sec}_3(C)$.

Set x_1 to be the class of X_1. From the above we have $(p_2)_*(x_1) = [\mathrm{Sec}_3(C)]$. Let $h' \subseteq \mathbf{P}^{d-2}$ be a hyperplane class and set $h := (p_2)^*(h') \subseteq \mathrm{Pic}^3(C) \times \mathbf{P}^{d-2}$.
Since $\mathrm{Sec}_3(C) \subseteq \mathbf{P}^{d-2}$ has dimension 5, we obtain the degree of $\mathrm{Sec}_3(C)$ by intersecting with $(h')^5$.
Now we have the following:

$$\begin{aligned} \deg(\mathrm{Sec}_3(C)) &= [Sec^3(C)].(h')^5 = (p_2)_*(x_1).(h')^5 \\ &= (p_2)_*(x_1.p_2^*(h')^5) = (p_2)_*(x_1.h^5) = x_1.h^5. \end{aligned}$$

That is, now we have to find the class x_1 of $X_1(\Phi)$.
Since $X_1(\Phi)$ has expected dimension $5 = \dim(\mathrm{Pic}^3(C) \times \mathbf{P}^{d-2}) - (d-4-1)(2-1)$, by Porteous' formula ([1], Chapter II, (4.2)) we obtain the following:

$$\begin{aligned} x_1 &= \Delta_{1,d-5}(c_t(F-E)) \\ &= \det \underbrace{\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{d-6} & c_{d-5} \\ 1 & c_1 & c_2 & \cdots & c_{d-7} & c_{d-6} \\ 0 & 1 & c_1 & \cdots & c_{d-8} & c_{d-7} \\ & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 1 & c_1 \end{pmatrix}}_{=:\mathcal{A}_{d-5}}, \end{aligned}$$

where $c_i := c_i(F-E)$, and $c_i(F-E)$ is defined via $c_t(F-E) := \frac{c_t(F)}{c_t(E)}$.
Again we use $\Theta_{P,Q}$ as defined in Section 2 when we talk about Θ on $\mathrm{Pic}^3(C) \cong \mathrm{Jac}(C)$.
Equations (1) and (2) gave us the following Chern polynomials:

$$c_t(\mathcal{H}) = c_t(\mathcal{G}) = e^{-\Theta t}.$$

We thus obtain

$$c_t(F) = c_t(p_1^*\mathcal{H}^*) = c_{-t}(p_1^*\mathcal{H}) = e^{p_1^*\Theta t}.$$

We compute $c_t(E)$:
Let α_i be the Chern roots of $\mathcal{G}$, i.e. $c_t(\mathcal{G}) = \prod_{i=1}^{d-4}(1+\alpha_i t)$, and set $\beta_i := p_1^*(\alpha_i)$.
Then we obtain the following:

$$
\begin{aligned}
c_t(E) &= \prod_{i=1}^{d-4}(1+(\beta_i-h)\,t) = \prod_{i=1}^{d-4}(1-ht)\left(1+\beta_i\frac{t}{1-ht}\right) \\
&= (1-ht)^{d-4}\prod_{i=1}^{d-4}\left(1+\beta_i\frac{t}{1-ht}\right) = (1-ht)^{d-4}c_{\frac{t}{1-ht}}\left(p_1^*\mathcal{G}\right) \\
&= (1-ht)^{d-4}e^{\frac{-p_1^*\Theta t}{1-ht}}.
\end{aligned}
$$

In the following we will identify Θ with $p_1^*(\Theta)$, it will be clear from the context if we mean Θ on $\mathrm{Pic}^3(C)$ or Θ on $\mathrm{Pic}^3(C)\times\mathbf{P}^{d-2}$.
We conclude now:

$$
\begin{aligned}
c_t(F-E) &= e^{\Theta t}(1-ht)^{4-d}e^{\frac{\Theta t}{1-ht}} = (1-ht)^{4-d}e^{\frac{2\Theta t-\Theta.ht^2}{1-ht}} \\
&= (1-ht)^{4-d}\sum_{j=0}^{\infty}\frac{1}{j!}(2\Theta t-\Theta.ht^2)^j(1-ht)^{-j} \\
&= \sum_{j=0}^{\infty}(1-ht)^{4-d-j}\frac{1}{j!}(2\Theta t-\Theta.ht^2)^j.
\end{aligned}
$$

Since $\Theta^3=0$, we only get some contribution from $j=0,1,2$ and thus obtain the following:

$$
\begin{aligned}
c_t(F-E) &= (1-ht)^{4-d}+(1-ht)^{3-d}(2\Theta t-\Theta.ht^2) \\
&+ \frac{1}{2}(1-ht)^{2-d}(4\Theta^2t^2-4\Theta^2.ht^3+\Theta^2.h^2t^4) \\
&= \sum_{k=0}^{\infty}\binom{d+k-3}{k}h^kt^k \\
&+ \sum_{k=0}^{\infty}\binom{d+k-3}{k}(2\Theta.h^k-2h^{k+1})t^{k+1} \\
&+ \sum_{k=0}^{\infty}\binom{d+k-3}{k}(2\Theta^2.h^k-3\Theta.h^{k+1}+h^{k+2})t^{k+2} \\
&+ \sum_{k=0}^{\infty}\binom{d+k-3}{k}(\Theta.h^{k+2}-2\Theta^2.h^{k+1})t^{k+3} \\
&+ \sum_{k=0}^{\infty}\frac{1}{2}\binom{d+k-3}{k}\Theta^2.h^{k+2}t^{k+4}.
\end{aligned}
$$

This implies that

$$\begin{aligned}
c_i(F-E) &= \left(\binom{d-3+i}{i} - 2\binom{d-4+i}{i-1} + \binom{d-5+i}{i-2}\right) h^i \\
&+ \left(2\binom{d-4+i}{i-1} - 3\binom{d-5+i}{i-2} + \binom{d-6+i}{i-3}\right) \Theta.h^{i-1} \\
&+ \left(2\binom{d-5+i}{i-2} - 2\binom{d-6+i}{i-3} + \frac{1}{2}\binom{d-7+i}{i-4}\right) \Theta^2.h^{i-2} \\
&= \binom{d-5+i}{i} h^i \\
&+ \left(\binom{d-5+i}{i-1} + \binom{d-6+i}{i-1}\right) \Theta.h^{i-1} \\
&+ \left(2\binom{d-6+i}{i-2} + \frac{1}{2}\binom{d-7+i}{i-4}\right) \Theta^2.h^{i-2}.
\end{aligned}$$

Now the last step in the computation of x_1 is to find the determinant of the matrix $\mathcal{A}_{d-5}$:

Proposition 5.1. *Set $c_i := c_i(F-E)$, where $c_i(F-E)$ is defined via $c_t(F-E) := \frac{c_t(F)}{c_t(E)}$. For $d \geq 8$ the determinant of the matrix*

$$\mathcal{A}_{d-5} = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{d-6} & c_{d-5} \\ 1 & c_1 & c_2 & \cdots & c_{d-7} & c_{d-6} \\ 0 & 1 & c_1 & \cdots & c_{d-8} & c_{d-7} \\ & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 1 & c_1 \end{pmatrix}$$

is equal to

$$\begin{aligned}
\mathcal{D}_{d-5} &= \left(\frac{1}{2}\binom{d-2}{3} - (d-4)\right) \Theta^2.h^{d-7} \\
&+ \left(\binom{d-3}{2} - 1\right) \Theta.h^{d-6} \\
&+ (d-4)h^{d-5}.
\end{aligned}$$

Proof. Let d be fixed. Set $d_0 := 1$ and for $n = 1, \ldots, d-5$, $k = 2, \ldots d-6$, set

$$d_n := \det \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ 1 & c_1 & c_2 & \cdots & c_{n-1} \\ & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & \cdots & 1 & c_1 \end{pmatrix}$$

and

$$b_{n,k} := \det \begin{pmatrix} c_k & c_{k+1} & c_{k+2} & \cdots & c_{n-1} & c_n \\ 1 & c_1 & c_2 & \cdots & c_{n-(k+1)} & c_{n-k} \\ 0 & 1 & c_1 & \cdots & c_{n-(k+2)} & c_{n-(k+1)} \\ & & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 1 & c_1 \end{pmatrix}.$$

By expansion with respect to the first column we have for each n and k:

$$d_n \;=\; c_1 d_{n-1} - b_{n,2}$$

and

$$b_{n,k} \;=\; c_k d_{n-k} - b_{n,k+1}.$$

This gives us by induction:

$$d_n \;=\; \sum_{i=1}^{n} (-1)^{i-1} c_i d_{n-i}.$$

Computing d_n for low n leads us to the following statement:

Lemma 5.2. *For $n \geq 3$ we have*

$$\begin{aligned} d_n \;&=\; \binom{d-4}{n} h^n + \left(\binom{d-3}{n} - \binom{d-5}{n} \right) \Theta . h^{n-1} \\ &+\; \left(\frac{1}{2}\binom{d-2}{n} - \binom{d-4}{n} + \frac{1}{2}\binom{d-6}{n} \right) \Theta^2 . h^{n-2}. \end{aligned}$$

Proof. By induction over n:

$$
\begin{aligned}
d_n &= \sum_{i=1}^{n}(-1)^{i-1}c_i d_{n-i} \\
&= \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i}\binom{d-4}{n-i}h^n \\
&+ \sum_{i=1}^{n}(-1)^{i-1}\Bigg(\binom{d-5+i}{i}\binom{d-3}{n-i}-\binom{d-5+i}{i}\binom{d-5}{n-i} \\
&\qquad +\binom{d-4}{n-i}\binom{d-6+i}{i-1}+\binom{d-4}{n-i}\binom{d-5+i}{i-1}\Bigg)\Theta.h^{n-1} \\
&+ \sum_{i=1}^{n}(-1)^{i-1}\Bigg(\frac{1}{2}\binom{d-5+i}{i}\binom{d-2}{n-i}-\binom{d-5+i}{i}\binom{d-4}{n-i} \\
&\qquad +\frac{1}{2}\binom{d-5+i}{i}\binom{d-6}{n-i}+\binom{d-6+i}{i-1}\binom{d-3}{n-i} \\
&\qquad -\binom{d-6+i}{i-1}\binom{d-5}{n-i}+\binom{d-5+i}{i-1}\binom{d-3}{n-i} \\
&\qquad -\binom{d-5+i}{i-1}\binom{d-5}{n-i}+2\binom{d-6+i}{i-2}\binom{d-4}{n-i} \\
&\qquad +\frac{1}{2}\binom{d-7+i}{i-4}\binom{d-4}{n-i}\Bigg)\Theta^2.h^{n-2}.
\end{aligned}
$$

Using the binomial identities

(a) Upper negation: $\binom{-r}{m} = (-1)^m\binom{r+m-1}{m}$ for $r, m \in \mathbf{N}$,
(b) Vandermonde's identity: $\sum_{k=0}^{r}\binom{m}{k}\binom{s}{r-k} = \binom{m+s}{r}$ for $m, r, s \in \mathbf{N}$

we obtain the formula for d_n as given in Lemma 5.2. $\square$

To finish the proof of Proposition 5.1 we use Lemma 5.2 taking $n = d-5 \geq 3$:

$$\begin{aligned}
\mathcal{D}_{d-5} &= d_{d-5} = \binom{d-4}{d-5} h^{d-5} \\
&+ \left(\binom{d-3}{d-5} - \binom{d-5}{d-5} \right) \Theta . h^{d-6} \\
&+ \left(\frac{1}{2}\binom{d-2}{d-5} - \binom{d-4}{d-5} + \frac{1}{2}\binom{d-6}{d-5} \right) \Theta^2 . h^{d-7} \\
&= (d-4) h^{d-5} \\
&+ \left(\binom{d-3}{2} - 1 \right) \Theta . h^{d-6} \\
&+ \left(\frac{1}{2}\binom{d-2}{3} - (d-4) \right) \Theta^2 . h^{d-7}.
\end{aligned}$$

□

Now we are able to deduce the formula for the degree of $\mathrm{Sec}_3(C)$ where C is a curve of genus 2 and degree $d \geq 8$ in $\mathbf{P}^{d-2}$:

Proposition 5.3. *The degree of the third secant variety* $\mathrm{Sec}_3(C)$ *of a smooth curve of genus* 2 *and degree* $d \geq 8$ *in* $\mathbf{P}^{d-2}$ *is equal to*

$$\binom{d-2}{3} - 2(d-4).$$

Proof. Since $\mathrm{Sec}_3(C)$ has dimension 5, we have to intersect with $(h')^5$ where h' is a hyperplane class in $\mathbf{P}^{d-2}$ in order to obtain the degree of $\mathrm{Sec}_3(C)$. From the above remarks we now have to find $\deg x_1 . h^5$, where $h = (p_2)^*(h')$.
We have

$$\begin{aligned}
\deg x_1 . h^5 &= \deg \mathcal{D}_{d-5} . h^5 \\
&= \deg \left(\frac{1}{2}\binom{d-2}{3} - (d-4) \right) \Theta^2 . h^{d-2}.
\end{aligned}$$

Since $\deg \Theta^2 . h^{d-2} = 2$, (cf. Proposition 2.2) we finally obtain

$$\deg \mathcal{D}_{d-5} . h^5 = 2\left(\frac{1}{2}\binom{d-2}{3} - (d-4) \right) = \binom{d-2}{3} - 2(d-4).$$

□

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References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris: *Geometry of Algebraic Curves*, Springer-Verlag, 1985.

[2] E. Ballico and A. Cossidente: Surfaces in $\mathbf{P}^5$ which do not admit trisecants, *Rocky Mountain Journal of Mathematics* **29(1)** (1999) 77–91.

[3] D. Eisenbud and J. Harris: On Varieties of Minimal Degree (A Centennial Account), *Proceedings of Symposia in Pure Mathematics* **46(1)** (1987) 3–13.

[4] W. Fulton: *Intersection Theory*, Springer-Verlag, 1998.

[5] R. Hartshorne: *Algebraic Geometry*, Springer-Verlag, 1977.

[6] J. S. Milne: Jacobian Varieties. In G. Cornell, J. H. Silverman (editors): *Arithmetic Geometry*, Springer-Verlag, 1986, 167–212.

Matematisk Institutt, Universitetet I Oslo, PO Box 1053, Blindern, NO-0316 Oslo, Norway

E-mail address: andrehof@math.uio.no